

Poisson loglinear modeling with linear constraints on the expected cell frequencies*

Nirian Martín

Dep. Statistics, Carlos III University of Madrid

Leandro Pardo

Dep. Statistics and O.R., Complutense University of Madrid

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Abstract

In this paper we consider Poisson loglinear models with linear constraints (LMLC) on the expected table counts. Multinomial and product multinomial loglinear models can be obtained by considering that some marginal totals (linear constraints on the expected table counts) have been prefixed in a Poisson loglinear model. Therefore with the theory developed in this paper, multinomial and product multinomial loglinear models can be considered as a particular case. To carry out inferences on the parameters in the LMLC an information-theoretic approach is followed from which the classical maximum likelihood estimators and Pearson chi-square statistics for goodness-of fit are obtained. In addition, nested hypotheses are proposed as a general procedure for hypothesis testing. Through a simulation study the appropriateness of proposed inference tools is illustrated.

Keywords: Loglinear Model, Marginal Model, Sampling Scheme, Restricted Estimators, Phi-divergence Measures.

1 Introduction

We consider a contingency table with k cells $\mathbf{n} = (n_1, \dots, n_k)^T$, with n_i being the observed frequency associated with the i -th cell ($i = 1, \dots, k$), its distribution is given by a Poisson random variable and since all of them are mutually independent the joint distribution of the contingency table is totally specified. Through a loglinear model $\log \mathbf{m}(\boldsymbol{\theta}) = \mathbf{X}\boldsymbol{\theta}$ a pattern is established for the mean vector of the contingency table, $\mathbf{m}(\boldsymbol{\theta}) \equiv (m_1(\boldsymbol{\theta}), \dots, m_k(\boldsymbol{\theta}))^T$, $m_i(\boldsymbol{\theta}) = E[n_i]$, $i = 1, \dots, k$, where \mathbf{X} is a known $k \times t$ full rank design matrix such that $t \leq k$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_t)^T \in \mathbb{R}^t$ is the vector of unknown parameters of the loglinear model.

Let

$$\mathcal{C}(\mathbf{X}) \equiv \{\log \mathbf{m}(\boldsymbol{\theta}) : \log \mathbf{m}(\boldsymbol{\theta}) = \mathbf{X}\boldsymbol{\theta}; \boldsymbol{\theta} \in \mathbb{R}^t\}$$

be the range of loglinear models associated with \mathbf{X} . We can observe that $\mathcal{C}(\mathbf{X})$ is the column space of matrix \mathbf{X} . A usual convention for loglinear models is to assume that the vector of 1's, $\mathbf{J}_k \equiv (1, \dots, 1)^T$, belongs to $\mathcal{C}(\mathbf{X})$, and therefore if a first column \mathbf{J}_k for \mathbf{X} is considered, the first term θ_1 of $\boldsymbol{\theta}$ is referred to the independent term of the model.

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In order to make statistical inference in the class of loglinear models $\mathcal{C}(\mathbf{X})$, Cressie and Pardo [7, 8] considered for the first time in loglinear models, minimum ϕ -divergence estimators and ϕ -divergence test-statistics. Later Martin and Pardo [22] presented a unified study for the three different sampling plans (multinomial, product-multinomial and Poisson).

To study some real situations on the basis of loglinear models, it is necessary to consider, in addition to loglinear models, some linear constraints. Loglinear models with linear constraints (LMLC) (see Definition 2.1) and product-multinomial sampling were considered for the first time by Haber and Brown [14]. One purpose of this paper is to consider divergence measures in order to make statistical inference (estimation and testing) in the class of LMLC but not only with product-multinomial sampling. We shall present a joint study for different sampling plans (multinomial, product-multinomial and Poisson). In addition, this article highlights the fact that the choice of additional lineal constraints is another way for nesting LMLC, in contrast to the traditional manner of nesting only log-linear constraints by reducing the number of columns of the design matrix. From this idea arises a new way for comparing LMLC that has not been previously considered in any paper and covers the preexistent hypothesis-testing techniques as special case (this point will be clarified in Section 4.2).

This article is organized as follows. In Section 2 we shall consider some notation as well as some preliminary concepts that will be important in the other sections of the paper. We pay special attention to the definition of phi-divergence measures between two non-negative vectors. Section 3 is devoted to define and study the minimum phi-divergence estimator of LMLC. The performed *constrained estimation method* will allow us to retain the advantage of dealing with Poisson loglinear models (specially to become estimation theory easier), even we could have, in fact, a multinomial or product-multinomial sampling plan. Moreover by an extension of such a method, if a marginal modeling itself is required, a compact estimation methodology is provided. As generalization of the constrained maximum likelihood estimation method, the constrained minimum ϕ -divergence estimation theory for LMLC is provided. Based also on ϕ -divergences, in Section 4 some test statistics for LMLC are proposed, specifically Section 4.1 is devoted to the problem of goodness-of-fit in LMLC and in Section 4.2 the problem of nested hypothesis in LMLC is studied. For both problems the asymptotic distribution of the ϕ -divergence test statistics under the null hypothesis are obtained. From such ϕ -divergence test statistics, in the case of the goodness-of-fit of LMLC, the classical likelihood ratio and Pearson chi-square test-statistics, presented in Haber and Brown [14] for multinomial and product-multinomial sampling schemes, are obtained as special case. In Section 5 three hypothesis tests, which share the aim for testing essentially a marginal model, are presented. The common framework of the LMLC developed in the previous cited sections, will allow us to carry out an easier comparison between them. An example of the potential versatility of such models will be shown in Remark 2.1, by considering apparently so different models, such as loglinear models and marginal models, within the same type of models. Some particular cases of loglinear models (symmetry, quasi-symmetry and ordinal quasi-symmetry) on one hand, and a marginal model on the other hand (marginal homogeneity model) are compared and the exact size and power of their hypothesis testing is analyzed.

2 Basic notation and definitions

By single index notation of \mathbf{n} we are able to unify a broad class of contingency tables, and by convention the terms of multiway contingency tables can be considered to be located in lexicographical order but by assigning a single index. For example in the usual double index notation for a two-way $I \times J$ contingency table $\mathbf{n} = (n_{11}, n_{12}, \dots, n_{1J}, \dots, n_{I1}, n_{I2}, \dots, n_{IJ})^T$, n_{ab} can be expressed by a one-to-one index transformation $i = (a - 1)J + b$, and therefore $k = IJ$. In a three-way $I \times J \times K$ contingency table, $\mathbf{n} = (n_{111}, n_{112}, \dots, n_{11K}, \dots, n_{1J1}, n_{1J2}, \dots, n_{1JK}, \dots, n_{I11}, n_{I12}, \dots, n_{I1K}, \dots, n_{IJ1}, n_{IJ2}, \dots, n_{IJK})^T$, n_{abc} can be expressed by a one-to-one index transformation $i = (a - 1)JK + (b - 1)K + c$, and therefore $k = IJK$. These models with single index notation are the so-called coordinate-free models (see Zelterman [31, Chapter 5]).

Product-multinomial sampling plan can be considered from a Poisson sampling plan with some additional linear constraints on the expected cell frequencies. Since c independent contingency subtables $\mathbf{n}_h = (n_{h1}, \dots, n_{hk_h})^T$ are considered in a product-multinomial sampling plan, the whole contingency table is $\mathbf{n} \equiv (\mathbf{n}_1^T, \dots, \mathbf{n}_c^T)^T$ and k is the summation of the number of cells in each subtable k_h , that is $k \equiv \sum_{h=1}^c k_h$. The marginal total in each subtable $\sum_{i=1}^{k_h} n_{hi} = \mathbf{J}_{k_h}^T \mathbf{n}_h$, $h = 1, \dots, c$ is prefixed to be $N_h \in \mathbb{N}$, and hence the mean vector $\mathbf{m}(\boldsymbol{\theta}) = (\mathbf{m}_1(\boldsymbol{\theta}), \dots, \mathbf{m}_c(\boldsymbol{\theta}))^T$ with an underlying Poisson sampling plan c linear constraints are verified

$$\mathbf{J}_{k_h}^T \mathbf{m}_h(\boldsymbol{\theta}) = \mathbf{J}_{k_h}^T \mathbf{n}_h, h = 1, \dots, c, \text{ or } \left(\bigoplus_{h=1}^c \mathbf{J}_{k_h}^T \right) \mathbf{m}(\boldsymbol{\theta}) = \left(\bigoplus_{h=1}^c \mathbf{J}_{k_h}^T \right) \mathbf{n}, \quad (1)$$

with $\bigoplus_{h=1}^d \mathbf{A}_d \equiv \text{diag}\{\mathbf{A}_1, \dots, \mathbf{A}_d\}$ representing the direct sum of d matrices. In particular, for multinomial sampling by taking $c = 1$ we have

$$\mathbf{J}_k^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{J}_k^T \mathbf{n}. \quad (2)$$

In what follows $c = 0$, i.e. the case where there is no any linear restriction associated with the sampling plan, will represent that the Poisson sampling itself is being taken into account.

It is well-known that there are some equivalences between the inferential results associated with the parameters for the three sampling plans (see for instance or instance in Lang [19, 20] and Agresti [1, Section 14.4]). The main reason why Poisson loglinear model is simpler to handle is based on the independence of the components of the sampling data.

The parameter space is given by

$$\Theta = \{\boldsymbol{\theta} \in \mathbb{R}^t : \mathbf{X}_0^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{X}_0^T \mathbf{n}\}, \quad (3)$$

where $\mathbf{X}_0 \equiv \bigoplus_{h=1}^c \mathbf{J}_{k_h}$ if $c \geq 1$ and \mathbf{X}_0 is a vector of zeros $\mathbf{0}_k$ if $c = 0$ (i.e., $\Theta = \mathbb{R}^t$). When $c \geq 2$ a stronger assumption than $\mathbf{J}_k \in \mathcal{C}(\mathbf{X})$ is taken into account, $\mathcal{C}(\mathbf{X}_0) \subset \mathcal{C}(\mathbf{X})$, and therefore if the first c columns \mathbf{X}_0 for \mathbf{X} are considered the h -th term θ_h ($h = 1, \dots, c$) of $\boldsymbol{\theta}$ is referred to the independent term for the model focussed only on the h -th contingency subtable.

Haber and Brown [14] considered multinomial and product multinomial LMLC but they did not consider the problem with Poisson sampling. Definition 2.1 is an extension of the definition given by Haber and Brown in which Poisson Loglinear models are included.

DEFINITION 2.1. *In addition of c linear constraints of the sampling scheme, consider $r \leq t - c$ linear constraints, $\mathbf{C}^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{d}^*$, i.e. \mathbf{C} and \mathbf{d}^* are $k \times r$ and $r \times 1$ matrices*

respectively. Once a loglinear model is established through a design matrix \mathbf{X} , a loglinear model with linear constraints is a simultaneous modeling of $\mathbf{m}(\boldsymbol{\theta})$ through a loglinear pattern on one hand and a linear pattern on the other hand

$$\log \mathbf{m}(\boldsymbol{\theta}) = \mathbf{X}\boldsymbol{\theta} \quad \text{and} \quad \mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{d}, \quad (4)$$

being $\mathbf{L} = (\mathbf{X}_0, \mathbf{C})$, $\mathbf{d} = (\mathbf{n}^T \mathbf{X}_0, (\mathbf{d}^*)^T)^T$, for $c \geq 1$, and $\mathbf{L} = \mathbf{C}$, $\mathbf{d} = \mathbf{d}^*$, for $c = 0$. It is also assumed to hold $k \geq t - c - r$, and $\text{rank}(\mathbf{L}) = \text{rank}(\mathbf{L}, \mathbf{d}) = c + r$.

Several examples are shown in Haber and Brown [14] for $c \geq 1$ and an application for $c = 0$ and $r \geq 1$ is suggested in Gail [13, Section 5] (for more details see Pardo and Martín [26]).

The parameter space of (4) is given by

$$\Theta = \{\boldsymbol{\theta} \in \mathbb{R}^t : \mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{d}\}. \quad (5)$$

It has been pointed-out that in most practical cases $\mathbf{d}^* = \mathbf{0}_r$, actually it holds in all examples of Haber and Brown [14]. Observe that $\mathbf{d}^* = (d_1^*, \dots, d_r^*)^T$ has been assumed to be constant, in fact if d_j^* ($j \in \{1, \dots, r\}$) is proportional to $N \equiv \sum_{i=1}^k m_i(\boldsymbol{\theta})$ there exists another equivalent constraint where $d_j^* = 0$.

In establishing asymptotic properties, we let N tend to infinity, and in this condition it is assumed that the normalized vector $\mathbf{m}^*(\boldsymbol{\theta}) = \mathbf{m}(\boldsymbol{\theta})/N$ remains fixed. For $c \geq 1$ this implies that, as $N \rightarrow \infty$, the probabilities in each cell remain fixed and N_h/N , $h = 1, \dots, c$, remain also fixed.

REMARK 2.1. *It is interesting to observe that we can consider two cases of LMLC:*

- i) The classical loglinear models without linear constraints, which are only defined through the loglinear pattern $\log \mathbf{m}(\boldsymbol{\theta}) = \mathbf{X}\boldsymbol{\theta}$ and thus $r = 0$ and $\mathbf{L} = \mathbf{X}_0$.*
- ii) The marginal models, which are only defined through the linear pattern $\mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{d}$ and thus by considering that \mathbf{X} is given by the identity matrix of order k , \mathbf{I}_k (i.e., $k = t$) the loglinear pattern is not itself a restriction.*

In the particular case $c = 0$ and $r = 0$, i.e., Poisson loglinear models without linear constraints, Cressie and Pardo [8] considered the problem of testing using divergence measures between probability vectors and solving the problem of estimation using the maximum likelihood estimator. Later in Martín and Pardo [22] the problem of estimation and testing was considered using divergence measures between nonnegative vectors but only for $r = 0$. Now in this paper the results obtained in Martín and Pardo [22] are extended for any $r \geq 0$. In this extension we consider the ϕ -divergence measure between nonnegative vectors.

Let Φ be the class of all convex and differentiable functions $\phi : [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$, such that at $x = 1$, $\phi(1) = \phi'(1) = 0$, $\phi''(1) > 0$. A ϕ -divergence measure between the \mathbb{R}_+^k -vectors $\mathbf{a} = (a_1, \dots, a_k)^T$ and $\mathbf{b} = (b_1, \dots, b_k)^T$ is given by

$$D_\phi(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^k b_i \phi\left(\frac{a_i}{b_i}\right), \quad \phi \in \Phi, \quad (6)$$

where $0\phi(0/0) \equiv 0$ and $0\phi(p/0) \equiv p \lim_{u \rightarrow \infty} \phi(u)/u$ conventions are assumed. These measures cover the traditional ones for probabilistic arguments, analyzed in Pardo [24], and all of them share similar properties. In particular by taking $\lambda \in \mathbb{R}$ and

$$\phi_{(\lambda)}(x) = \frac{x^{\lambda+1} - x - \lambda(x-1)}{\lambda(\lambda+1)}, \quad \text{if } \lambda(\lambda+1) \neq 0, \quad (7)$$

and $\phi_{(\lambda^*)}(x) = \lim_{\lambda \rightarrow \lambda^*} \phi_{(\lambda)}(x)$, if $\lambda^* \in \{0, -1\}$, power divergence measures, introduced in Cressie and Read [9], are obtained. The so-called Kullback divergence measure is obtained through $\phi_{(0)}(x) = x \log x - x + 1$,

$$D_{Kull}(\mathbf{a}, \mathbf{b}) \equiv D_{\phi_{(0)}}(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^k a_i \log \left(\frac{a_i}{b_i} \right) - \sum_{i=1}^k a_i + \sum_{i=1}^k b_i, \quad (8)$$

which was given between two non-negative vectors for the first time in Brockett [4]. It should be pointed out that the way in which asymptotic results were obtained in [22] is primarily focussed on the parameter vector, being the mean vector a secondary aim, and therefore this way is just opposite to the one followed for other works related to loglinear modeling (see for instance Lang [18]) where the primary aim is the mean vector itself. These measures and also the methodology for developing asymptotic results will remain being useful for obtaining the asymptotic results associated with LMLC. Taking into account Remark 2.1, it is important to clarify that apart from the possibility of reproducing all inferential results obtained previously in Martín and Pardo [22], the new results of this paper are important because the LMLC cover a broad range of models.

3 Minimum ϕ -divergence estimator

The maximum likelihood estimator (MLE) $\hat{\boldsymbol{\theta}}$ of the parameter in (4) can be obtained by maximizing the kernel of the Poisson log-likelihood

$$\ell(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})) \equiv \sum_{i=1}^k n_i \log m_i(\boldsymbol{\theta}) - \sum_{i=1}^k m_i(\boldsymbol{\theta}),$$

subject to the constraints $\mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}) = \mathbf{d}$, i.e. on the basis of (5)

$$\hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta} \in \Theta} \ell(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})).$$

Observe that according to Definition 2.1, if $\mathbf{X}_0 = \bigoplus_{h=1}^c \mathbf{J}_{k_h}$, which takes part in \mathbf{L} as submatrix, the underlying sampling plan is product-multinomial (or multinomial, if $c = 1$). In what follows even sometimes (product) multinomial sampling will not be explicitly mentioned, in all results this sampling plan will be covered.

On the basis of (8) we have

$$D_{Kull}(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})) = \sum_{i=1}^k n_i \log n_i - \sum_{i=1}^k n_i - \ell(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})),$$

and it is possible also define the MLE of the parameter in (4) by

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta} \in \Theta} D_{Kull}(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})).$$

Rather than using MLE, one could use divergence based methods for estimating the parameters of the loglinear models with linear constraints. On the basis of (6) a minimum ϕ -divergence estimator (M ϕ E) for a LMLC, given in Definition 2.1, is defined as follows.

DEFINITION 3.1. *For a LMLC (4) with parameter space (5), the M ϕ E is given by*

$$\hat{\boldsymbol{\theta}}^\phi = \arg \min_{\boldsymbol{\theta} \in \Theta} D_\phi(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta})), \quad (9)$$

with $D_\phi(\mathbf{n}, \mathbf{m}(\boldsymbol{\theta}))$ defined by (6).

In Aitchison and Silvey [3] a method for finding MLE's subject to constraints and its asymptotic distribution theory was developed for the first time using the Lagrange multiplier method. In Pardo et al. [25] a MφE procedure for multinomial models was introduced in which the probabilities depend on unknown parameters that satisfy some functional relationships. Following the last method but more generally in the sense that the probabilities are replaced by means, in the following theorem we present the key result for developing the asymptotic distribution theory for LMLC, the decomposition of the MφE for the parameter vector.

THEOREM 3.1. *Suppose that the data $\mathbf{n} = (n_1, \dots, n_k)^T$ are Poisson distributed whose mean vector is given by a LMLC (4). Choosing a function $\phi \in \Phi$, where Φ was defined in Section 1, we have*

$$\hat{\boldsymbol{\theta}}^\phi = \boldsymbol{\theta}_0 + \mathbf{H}(\boldsymbol{\theta}_0) \mathbf{X}^T \left(\frac{\mathbf{n}}{N} - \mathbf{m}^*(\boldsymbol{\theta}_0) \right) + o \left(\left\| \frac{\mathbf{n}}{N} - \mathbf{m}^*(\boldsymbol{\theta}_0) \right\| \right),$$

where

$$\begin{aligned} \mathbf{H}(\boldsymbol{\theta}_0) &\equiv \mathbf{I}_{\mathcal{F}}(\boldsymbol{\theta}_0)^{-1} - \mathbf{I}_{\mathcal{F}}(\boldsymbol{\theta}_0)^{-1} \mathbf{B}(\boldsymbol{\theta}_0) \left(\mathbf{B}(\boldsymbol{\theta}_0)^T \mathbf{I}_{\mathcal{F}}(\boldsymbol{\theta}_0)^{-1} \mathbf{B}(\boldsymbol{\theta}_0) \right)^{-1} \\ &\quad \times \mathbf{B}(\boldsymbol{\theta}_0)^T \mathbf{I}_{\mathcal{F}}(\boldsymbol{\theta}_0)^{-1}, \\ \mathbf{I}_{\mathcal{F}}(\boldsymbol{\theta}_0) &\equiv \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)} \mathbf{X} \text{ (Fisher information matrix associated with the} \\ &\quad \text{Poisson loglinear model),} \\ \mathbf{B}(\boldsymbol{\theta}_0) &\equiv \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)} \mathbf{L}, \end{aligned}$$

$\mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}$ is the diagonal matrix of the normalized vector $\mathbf{m}^*(\boldsymbol{\theta})$ and $\boldsymbol{\theta}_0 \in \Theta$ is the true and unknown value of the parameter.

Proof. We omit the proof because its steps are similar to ones given in Martín and Pardo [21] with the differences motivated because in the cited paper only multinomial sampling was considered. \square

In the next theorem we obtain the asymptotic distribution of $\hat{\boldsymbol{\theta}}^\phi$ as well as of $\mathbf{m}(\hat{\boldsymbol{\theta}}^\phi)$.

THEOREM 3.2. *Suppose that the data $\mathbf{n} = (n_1, \dots, n_k)^T$ are Poisson distributed whose mean vector is given by a LMLC (4). Choosing a function $\phi \in \Phi$, where Φ was defined in Section 1, we have*

a)

$$\sqrt{N}(\hat{\boldsymbol{\theta}}^\phi - \boldsymbol{\theta}_0) \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_t, \mathbf{H}(\boldsymbol{\theta}_0)) \quad (10)$$

where $\mathbf{H}(\boldsymbol{\theta}_0)$ is defined in Theorem 3.1, " $\xrightarrow[N \rightarrow \infty]{\mathcal{L}}$ " denotes convergence in law (or distribution)

and

b)

$$\frac{1}{\sqrt{N}}(\mathbf{m}(\hat{\boldsymbol{\theta}}^\phi) - \mathbf{m}(\boldsymbol{\theta}_0)) \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_k, \boldsymbol{\Sigma}) \quad (11)$$

where

$$\boldsymbol{\Sigma} \equiv \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)} \mathbf{X} \mathbf{H}(\boldsymbol{\theta}_0) \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}.$$

Proof. Result a) follows by Theorem 3.1 and taking into account (see Haberman [16])

$$\frac{1}{\sqrt{N}}(\mathbf{n} - \mathbf{m}(\boldsymbol{\theta}_0)) \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_k, \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}). \quad (12)$$

Part b) follows by a) and applying delta method (see for instance Agresti [1, Sections 14.1.2, 14.1.3]). \square

In the next theorem a result related to a simplification of the expression of the asymptotic variance-covariance matrices of Theorem 3.2 is shown.

THEOREM 3.3. *When*

$$\mathbf{X} = (\mathbf{L}, \mathbf{W}) \quad (13)$$

we have

$$\mathbf{H}(\theta_0) = (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - (\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L})^{-1} \oplus \mathbf{0}_{(t-c-r) \times (t-c-r)}$$

and

$$\Sigma = \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}} (\mathbf{A}_X(\theta_0) - \mathbf{A}_L(\theta_0)) \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}},$$

where

$$\begin{aligned} \mathbf{A}_X(\theta_0) &\equiv \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}} \mathbf{X} (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}, \\ \mathbf{A}_L(\theta_0) &\equiv \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}} \mathbf{L} (\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L})^{-1} \mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}. \end{aligned}$$

We can observe that $\mathbf{A}_X(\theta_0)$ and $\mathbf{A}_L(\theta_0)$ are projector matrices respectively on column spaces $\mathcal{C}(\mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}} \mathbf{X})$ and $\mathcal{C}(\mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}} \mathbf{L})$.

Proof. By starting through a identity matrix,

$$\begin{aligned} \mathbf{I}_t &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X} \\ &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} (\mathbf{L}, \mathbf{W}) \\ &= \left((\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L}, (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{W} \right), \end{aligned}$$

it is obtained that

$$\mathbf{G}_{t \times (c+r)} = (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L} = \begin{pmatrix} \mathbf{I}_{c+r} \\ \mathbf{0}_{(t-c-r) \times (c+r)} \end{pmatrix}.$$

Therefore

$$\begin{aligned} \mathbf{H}(\theta_0) &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L} \\ &\quad \times \left(\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X} (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L} \right)^{-1} \\ &\quad \times \mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X} (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} \\ &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - \mathbf{G}_{t \times (c+r)} (\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} (\mathbf{L}, \mathbf{W}) \mathbf{G}_{t \times c}^T)^{-1} \mathbf{G}_{t \times (c+r)}^T \\ &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - \mathbf{G}_{t \times (c+r)} (\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L})^{-1} \mathbf{G}_{t \times (c+r)}^T \\ &= (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - (\mathbf{L}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{L})^{-1} \oplus \mathbf{0}_{(t-c-r) \times (t-c-r)}. \end{aligned}$$

On the other hand, the expression of Σ is obtained replacing the new expression of $\mathbf{H}(\theta_0)$ inside its original definition in Theorem 3.2. \square

REMARK 3.1. *When $r = 0$ and $c = 1$ (classical multinomial loglinear model), $\mathbf{X} = (\mathbf{J}_k, \mathbf{W})$, $\mathbf{J}_k^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{J}_k = \mathbf{1}$ and thus it holds $\mathbf{H}(\theta_0) = (\mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\theta_0)} \mathbf{X})^{-1} - \mathbf{1} \oplus \mathbf{0}_{(t-1) \times (t-1)}$ as direct application of Theorem 3.3. If we pay attention to the structure of this variance-covariance matrix, we can observe that the $\boldsymbol{\theta} = (\theta_1, \bar{\boldsymbol{\theta}}^T)^T$ is partitioned*

in such a way that once the part associated with \mathbf{W} , $\bar{\boldsymbol{\theta}} = (\theta_2, \dots, \theta_t)^T$, is known, the first component θ_1 can be obtained through $\bar{\boldsymbol{\theta}}$ and the linear constraint. This is the reason why in the traditional multinomial loglinear modeling the dimension of the parameter space is $t - 1$ instead of t and $\theta_1 = N/(\mathbf{J}_k^T \exp\{\mathbf{W}\bar{\boldsymbol{\theta}}\})$ is the redundant component of the multinomial loglinear model. When $c \geq 0$ and $r \geq 1$, it is possible to partitionate and interpret any parameter vector $\boldsymbol{\theta}$ in terms of (13). Due to space limitation, we omit it in a formal way. In a less formal way we can say that making transformation on the design or restrictions matrices, it is possible to obtain LMLC with an structure for the design matrix like in (13). The part of the parameters associated with matrix \mathbf{W} , are “free parameters”, while the rest of the terms are determinated through a function. It is frequent to find textbooks that consider only free parameters for making statistical inferences.

4 ϕ -divergence test statistics

4.1 Goodness-of-fit

Classical measures for assessing the goodness-of-fit of categorical data models, estimated by MLE, are the likelihood ratio test statistic, sometimes referred to as the deviance statistic,

$$G^2(\hat{\boldsymbol{\theta}}) = 2 \sum_{i=1}^k \left(n_i \log \frac{n_i}{m_i(\hat{\boldsymbol{\theta}})} - (n_i - m_i(\hat{\boldsymbol{\theta}})) \right), \quad (14)$$

and Pearson chi-square test statistic

$$X^2(\hat{\boldsymbol{\theta}}) = \sum_{i=1}^k \frac{(n_i - m_i(\hat{\boldsymbol{\theta}}))^2}{m_i(\hat{\boldsymbol{\theta}})}. \quad (15)$$

In Haber and Brown [14] the asymptotic distribution of a classical goodness-of-fit test-statistics for LMLC when the sampling scheme is (product) multinomial ($c \geq 1$, $r \geq 0$) was established. On the other hand in Martín and Pardo [22] divergence based goodness-of-fit test-statistics were analyzed for loglinear models under the three sampling schemes ($c \geq 0$) when none constraint additional to the sampling ones are considered ($r = 0$). In this section we extend the previous result to the important context in which $r > 0$. In this framework the family of ϕ -divergence test statistics is given by

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2}) = \frac{2}{\phi_1''(1)} D_{\phi_1}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}^{\phi_2})). \quad (16)$$

Observe that while the divergence based estimator is associated with a specific ϕ_2 function, the divergence based test-statistic is associated with a function ϕ_1 , not necessarily equal to ϕ_2 , in fact while $G^2(\hat{\boldsymbol{\theta}}) = T^{\phi_{(0)}}(\hat{\boldsymbol{\theta}}^{\phi_{(0)}})$ where $\phi_{(0)}(x) = x \log x - x + 1$, it holds $X^2(\hat{\boldsymbol{\theta}}) = T^{\phi_{(1)}}(\hat{\boldsymbol{\theta}}^{\phi_{(0)}})$ where $\phi_{(1)}(x) = \frac{1}{2}(x - 1)^2$.

In the following theorem we establish that the asymptotic distribution of the family of ϕ -divergence test statistics, $T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2})$, is a chi-square with $k - t + c$ degrees of freedom (χ_{k-t+c}^2). Therefore, we do not accept the null hypothesis in which the model is said to be (4) if $T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2}) > c$, where c is specified so that the size of the test is α , $\Pr(T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2}) > c \mid H_{Null}) = \Pr(\chi_{k-t+c}^2 > c) = \alpha$, i.e. $c \equiv \chi_{k-t+c}^2(1 - \alpha)$ is the $(1 - \alpha)$ -th quantile of a χ_{k-t+c}^2 distribution.

THEOREM 4.1. *Suppose that the data $\mathbf{n} = (n_1, \dots, n_k)^T$ are Poisson distributed. Choose*

the function $\phi_1, \phi_2 \in \Phi$, where Φ was defined in Section 1. Then, for testing

$$\begin{aligned} H_{Null} \log \mathbf{m}(\boldsymbol{\theta}) \in \mathcal{C}(\mathbf{X}) \text{ and } \boldsymbol{\theta} \in \Theta = \{\boldsymbol{\theta}' \in \mathbb{R}^t : \mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}') = \mathbf{d}\}, \\ H_{Alt} \log \mathbf{m}(\boldsymbol{\theta}) \notin \mathcal{C}(\mathbf{X}) \text{ or } \boldsymbol{\theta} \notin \Theta = \{\boldsymbol{\theta}' \in \mathbb{R}^t : \mathbf{L}^T \mathbf{m}(\boldsymbol{\theta}') = \mathbf{d}\}, \end{aligned} \quad (17)$$

the asymptotic null distribution of the test statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2})$, given in (16), is chi-squared with $k - t + c$ degrees of freedom.

Proof. We consider the function $f(x, y) = x\phi_1(y/x)$. A second order Taylor's expansion of $f(\frac{n_i}{N}, m_i^*(\hat{\boldsymbol{\theta}}^{\phi_2}))$ about $(m_i^*(\boldsymbol{\theta}_0), m_i^*(\boldsymbol{\theta}_0))$ gives

$$f\left(\frac{n_i}{N}, m_i^*(\hat{\boldsymbol{\theta}}^{\phi_2})\right) = \frac{\phi_1''(1)}{2} \frac{\left(\frac{n_i}{N} - m_i^*(\hat{\boldsymbol{\theta}}^{\phi_2})\right)^2}{m_i^*(\boldsymbol{\theta}_0)} + o_P(N^{-1}); \quad i = 1, \dots, k.$$

Taking into account

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2}) = \frac{2N}{\phi_1''(1)} \sum_{i=1}^k f\left(\frac{n_i}{N}, m_i^*(\hat{\boldsymbol{\theta}}^{\phi_2})\right) = \sum_{i=1}^k Z_i^2 + o_P(1),$$

where

$$Z_i \equiv \frac{n_i - m_i(\hat{\boldsymbol{\theta}}^{\phi_2})}{\sqrt{m_i(\boldsymbol{\theta}_0)}}, \quad i = 1, \dots, k,$$

we obtain the following vectorial expression

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2}) = \mathbf{Z}^T \mathbf{Z} + o_P(1),$$

being

$$\mathbf{Z} = (Z_1, \dots, Z_k)^T \equiv \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{-\frac{1}{2}} (\mathbf{n} - \mathbf{m}(\hat{\boldsymbol{\theta}}^{\phi_2})).$$

The random vector \mathbf{Z} is asymptotically normal distributed with mean vector zero and asymptotic variance-covariance matrix

$$\mathbf{T}^* \equiv \mathbf{I}_k - \mathbf{A}_0(\boldsymbol{\theta}_0) - \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}} \mathbf{X} \mathbf{H}(\boldsymbol{\theta}_0) \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}}, \quad (18)$$

where $\mathbf{A}_0(\boldsymbol{\theta}_0)$ is given by

$$\mathbf{A}_0(\boldsymbol{\theta}_0) = \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}} \mathbf{X}_0 (\mathbf{X}_0^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)} \mathbf{X}_0)^{-1} \mathbf{X}_0^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}}. \quad (19)$$

Then, the asymptotic distribution of the ϕ -divergence test statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2})$ will be a chi-square iff the matrix \mathbf{T}^* is idempotent and symmetric. It is clear that \mathbf{T}^* is symmetric, and to establish that it is idempotent we have

$$(\mathbf{T}^*)^2 = \mathbf{S}\mathbf{S} - \mathbf{S}\mathbf{K} - \mathbf{K}\mathbf{S} + \mathbf{K}\mathbf{K} = \mathbf{S} - \mathbf{K} - \mathbf{K} + \mathbf{K} = \mathbf{T}^*,$$

where $\mathbf{S} = \mathbf{I}_k - \mathbf{A}_0(\boldsymbol{\theta}_0)$ and $\mathbf{K} = \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}} \mathbf{X} \mathbf{H}(\boldsymbol{\theta}_0) \mathbf{X}^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_0)}^{\frac{1}{2}}$. The degrees of freedom of the chi-squared distributed statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}^{\phi_2})$ coincides with the trace of the matrix \mathbf{T}^* , i.e. $k - t + c$. \square

REMARK 4.1. When $\mathcal{C}(\mathbf{L}) \subset \mathcal{C}(\mathbf{X})$, because $\mathcal{C}(\mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}\mathbf{L}) \subset \mathcal{C}(\mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}\mathbf{X})$ it holds $\mathbf{A}_X(\theta_0)\mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}\mathbf{L} = \mathbf{D}_{\mathbf{m}^*(\theta_0)}^{\frac{1}{2}}\mathbf{L}$, which means that (18) is given by

$$\mathbf{T}^* = \mathbf{I}_k - \mathbf{A}_0(\theta_0) - \mathbf{A}_X(\theta_0) + \mathbf{A}_L(\theta_0). \quad (20)$$

From this expression it is concluded that when there is no any sampling constraint ($r = 0 \Rightarrow \mathbf{A}_L(\theta_0) = \mathbf{A}_0(\theta_0)$), the variance-covariance matrix (18) of the random vector \mathbf{Z} , under the assumption that the model of the null hypothesis in (17) holds, have a common expression, $\mathbf{T}^* = \mathbf{I}_k - \mathbf{A}_X(\theta_0)$, for the three sampling schemes ($c \geq 0$).

4.2 Nested hypothesis

Two models are said to be nested if one of them can be obtained from the other one as special case. This general definition for linear models (see Chatterjee and Hadi [6, page 65]) can be applied to two loglinear models, whose design matrices are given by \mathbf{X}_1 and \mathbf{X}_2 , in such a way that the first one is said to be nested within the second one if $\mathcal{C}(\mathbf{X}_1) \subset \mathcal{C}(\mathbf{X}_2)$. Observe that $\text{rank}(\mathbf{X}_1) = t_1 \leq \text{rank}(\mathbf{X}_2) = t_2$, and therefore $\Theta_1 = \{\theta' \in \mathbb{R}^{t_1} : \mathbf{X}_0^T \mathbf{m}(\theta') = \mathbf{X}_0^T \mathbf{n}\} \subset \Theta_2 = \{\theta' \in \mathbb{R}^{t_2} : \mathbf{X}_0^T \mathbf{m}(\theta') = \mathbf{X}_0^T \mathbf{n}\}$, i.e. $\forall \theta_1 \in \Theta_1 \exists \theta_2 \in \Theta_2 : \mathbf{m}(\theta_1) = \mathbf{m}(\theta_2)$. Moreover, if $\mathbf{X}_2 = (\mathbf{X}_1, \mathbf{Y}_2)$, where $\text{rank}(\mathbf{Y}_2) = s_2$ (i.e., $t_2 = t_1 + s_2$), by considering $\theta_2 = (\theta_1^T, \mathbf{0}_{s_2}^T)^T$, it holds $\mathbf{m}(\theta_1) = \mathbf{m}(\theta_2)$, and thus the loglinear model defined by \mathbf{X}_1 is nested within the loglinear model defined by \mathbf{X}_2 . In order to clarify that this is a particular case of nested model, a loglinear model defined by \mathbf{X}_1 is said to be a reduced loglinear model of $\mathbf{X}_2 = (\mathbf{X}_1, \mathbf{Y}_2)$.

In the following definition we consider a sequence of design matrices $\{\mathbf{X}_b\}_{b=1}^B$ so that the loglinear model associated with $\mathbf{X}_{b+1} = (\mathbf{L}, \mathbf{W}_{b+1})$ is a reduced loglinear model of $\mathbf{X}_b = (\mathbf{L}, \mathbf{W}_b)$, $b = 1, \dots, B-1$, which means that \mathbf{W}_{b+1} is a submatrix of \mathbf{W}_b . Such matrices define a sequence of LMLC that share the same linear constraints.

DEFINITION 4.1. *The sequence of LMLC*

$$\log \mathbf{m}(\theta_b) = \mathbf{X}_b \theta_b \quad \text{and} \quad \mathbf{L}^T \mathbf{m}(\theta_b) = \mathbf{d}, \quad (21)$$

where $\mathbf{X}_b = (\mathbf{x}_1, \dots, \mathbf{x}_{t-b+1})$, $b \in \{1, \dots, t-c-r\}$, is called the b -th reduced LMLC through the parameter, because by reducing one unit the dimension of the parameter space $\Theta_b \equiv \{\theta_b \in \mathbb{R}^{t-b+1} : \mathbf{L}^T \mathbf{m}(\theta_b) = \mathbf{d}\}$, it holds $M_{b+1} \subset M_b$ where

$$M_b \equiv \{\mathbf{m}(\theta_b) \in \mathbb{R}^k : \log \mathbf{m}(\theta_b) = \mathbf{X}_b \theta_b, \theta_b \in \Theta_b\}.$$

In the following definition we consider a sequence of constraints $\{\mathbf{L}_b \mathbf{m}(\theta) = \mathbf{d}_b\}_{b=1}^B$ so that the $(\mathbf{L}_b, \mathbf{d}_b)$ is a submatrix of $(\mathbf{L}_{b+1}, \mathbf{d}_{b+1})$. Such constraints define a sequence of LMLC that share the same design matrix $\mathbf{X} = (\mathbf{L}_b, \mathbf{W}_b)$.

DEFINITION 4.2. *The sequence of LMLC*

$$\log \mathbf{m}(\theta) = \mathbf{X} \theta \quad \text{and} \quad \mathbf{L}_b^T \mathbf{m}(\theta) = \mathbf{d}_b, \quad (22)$$

where $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_t)$, $\mathbf{L}_1 = (\mathbf{x}_1, \dots, \mathbf{x}_{c+r})$ and $\mathbf{L}_{b+1} = (\mathbf{L}_b, \mathbf{x}_{c+r+b})$, $b \in \{1, \dots, t-c-r\}$, is called the b -th reduced LMLC through the constraints, because by increasing one unit the number of constraints, since $\Theta_{b+1} \subset \Theta_b$ with the parameter space given by $\Theta_b \equiv \{\theta \in \mathbb{R}^t : \mathbf{L}_b^T \mathbf{m}(\theta) = \mathbf{d}_b\}$, it holds $M_{b+1} \subset M_b$ where

$$M_b \equiv \{\mathbf{m}(\theta) \in \mathbb{R}^k : \log \mathbf{m}(\theta) = \mathbf{X} \theta, \theta \in \Theta_b\}.$$

In the following a generalized definition of nested LMLC is given, in which Definitions 4.1 and 4.2 are covered.

DEFINITION 4.3. *In a sequence of LMLC $\{M_b\}_{b=1}^B$ such that*

$$\begin{aligned} M_b &\equiv \{\mathbf{m}(\boldsymbol{\theta}_b) \in \mathbb{R}^k : \log \mathbf{m}(\boldsymbol{\theta}_b) = \mathbf{X}_b \boldsymbol{\theta}_b, \boldsymbol{\theta}_b \in \Theta_b\}, \\ \Theta_b &\equiv \{\boldsymbol{\theta}_b \in \mathbb{R}^{t_b} : \mathbf{L}_b^T \mathbf{m}(\boldsymbol{\theta}_b) = \mathbf{d}_b\}, \\ t_b &\equiv \text{rank}(\mathbf{X}_b), \\ \mathbf{L}_b &\equiv (\mathbf{X}_0, \mathbf{C}_b), \\ r_b &\equiv \text{rank}(\mathbf{C}_b), \end{aligned}$$

M_{b+1} is said to be nested within M_b ($b \in \{1, \dots, B-1\}$), denoted by $M_{b+1} \subset M_b$, if it holds

$$\mathcal{C}(\mathbf{X}_{b+1}) \subset \mathcal{C}(\mathbf{X}_b) \quad \text{and} \quad \mathcal{C}(\mathbf{L}_b) \subset \mathcal{C}(\mathbf{L}_{b+1}), \quad (23)$$

with $t_{b+1} \leq t_b$ and $r_{b+1} \geq r_b$, being strict at least one of the two inequalities.

Once a sequence of nested LMLC $\{M_b\}_{b=1}^B$ has been established, our goal is to present ϕ -divergence test statistics to test successively

$$H_{Null}(b) : M_{b+1} \text{ against } H_{Alt}(b) : M_b - M_{b+1}; \quad b = 1, \dots, B-1, \quad (24)$$

where we continue to test as long as the null hypothesis is accepted and we infer an integer b^* , such that $b \in \{1, \dots, B-1\}$, to be the first value b for which M_{b+1} is rejected as null hypothesis, or $b^* = B$ otherwise.

In Agresti [1, Section 4.5.4] the classical likelihood ratio test statistic for loglinear models ($r_b = r_{b+1} = 0$, $c \geq 0$) is given,

$$\begin{aligned} G^2(\hat{\boldsymbol{\theta}}_{b+1} | \hat{\boldsymbol{\theta}}_b) &= 2 \sum_{i=1}^k \left(m_i(\hat{\boldsymbol{\theta}}_b) \log \frac{m_i(\hat{\boldsymbol{\theta}}_b)}{m_i(\hat{\boldsymbol{\theta}}_{b+1})} - m_i(\hat{\boldsymbol{\theta}}_b) + m_i(\hat{\boldsymbol{\theta}}_{b+1}) \right) \\ &= 2 \left(D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1})) - D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_b)) \right), \end{aligned} \quad (25)$$

where $\hat{\boldsymbol{\theta}}_{b+1}$ and $\hat{\boldsymbol{\theta}}_b$ are the MLE's of the parameter in the models M_{b+1} and M_b respectively. It is also shown that $G^2(\hat{\boldsymbol{\theta}}_{b+1} | \hat{\boldsymbol{\theta}}_b)$ is asymptotically distributed according to a chi-square with $t_b - t_{b+1}$ degrees of freedom under the null hypothesis of (24). Minimizing the Kullback divergence measure over a smaller parameter space cannot yield a larger minimum value, therefore $D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1})) \geq D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_b))$. However there is another interesting way to show the same inequality, which is based on proving

$$D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1})) - D_{Kull}(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_b)) = D_{Kull}(\mathbf{m}(\hat{\boldsymbol{\theta}}_b), \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1})), \quad (26)$$

whose non-negativity is guaranteed by the common property of all divergence measures. Based on (25) and (26), in Martín and Pardo [22, Section 4] divergence-based test statistics were introduced for the same models ($r_b = r_{b+1} = 0$, $c \geq 0$),

$$S^\phi(\hat{\boldsymbol{\theta}}_{b+1}^\phi | \hat{\boldsymbol{\theta}}_b^\phi) = \frac{2}{\phi''(1)} \left(D_\phi(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1}^\phi)) - D_\phi(\mathbf{n}, \mathbf{m}(\hat{\boldsymbol{\theta}}_b^\phi)) \right) \quad (27)$$

and

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \hat{\boldsymbol{\theta}}_b^{\phi_2}) = \frac{2}{\phi_1''(1)} D_{\phi_1}(\mathbf{m}(\hat{\boldsymbol{\theta}}_b^{\phi_2}), \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2})), \quad (28)$$

whose asymptotic distribution under the null hypothesis of (24) was shown to be exactly the same as $G^2(\hat{\boldsymbol{\theta}}_{b+1}|\hat{\boldsymbol{\theta}}_b)$ for both of them. It should be pointed out that (26) does not hold by replacing any ϕ -divergence measure instead of the Kullback divergence measure.

In the more general framework of LMLC ($r_{b+1} \geq r_b \geq 0$, $c \geq 0$) we shall establish herein that under the null hypothesis of (24), the test statistics $T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}|\hat{\boldsymbol{\theta}}_b^{\phi_2})$ and $S^{\phi}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi}|\hat{\boldsymbol{\theta}}_b^{\phi})$ converge in law to a chi-square with $t_b - t_{b+1} - r_b + r_{b+1}$ degrees of freedom ($\chi_{t_b - t_{b+1} - r_b + r_{b+1}}^2$), $b = 1, \dots, B-1$. Thus, $\chi_{t_b - t_{b+1} - r_b + r_{b+1}}^2(1 - \alpha)$ could be chosen as a cutpoint for the rejection region.

THEOREM 4.2. *Suppose that the data $\mathbf{n} = (n_1, \dots, n_k)^T$ are Poisson distributed. Choose the function $\phi_1, \phi_2 \in \Phi$, where Φ was defined in Section 1. Then, for testing (24), the asymptotic null distribution of the test statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}|\hat{\boldsymbol{\theta}}_b^{\phi_2})$, given in (28), is chi-squared with $t_b - t_{b+1} - r_b + r_{b+1}$ degrees of freedom.*

Proof. A similar Taylor's expansion to one given in Theorem 4.1 yields

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}|\hat{\boldsymbol{\theta}}_b^{\phi_2}) = \mathbf{Z}_b^T \mathbf{Z}_b + o_P(1),$$

where

$$\mathbf{Z}_b = D_{\mathbf{m}(\boldsymbol{\theta}_0)}^{-\frac{1}{2}}(\mathbf{m}(\hat{\boldsymbol{\theta}}_b^{\phi_2}) - \mathbf{m}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}))$$

is distributed asymptotically as a normal distribution with mean vector zero and variance-covariance matrix $\mathbf{T}_b^* = \mathbf{K}_b - \mathbf{K}_{b+1}$ with

$$\begin{aligned} \mathbf{K}_j \equiv & \mathbf{A}_{X_j}(\boldsymbol{\theta}_{b+1,0}) - \mathbf{A}_{X_j}(\boldsymbol{\theta}_{b+1,0}) D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{\frac{1}{2}} \mathbf{L}_j \left(\mathbf{L}_j^T D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{\frac{1}{2}} \right. \\ & \left. \times \mathbf{A}_{X_j}(\boldsymbol{\theta}_{b+1,0}) D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{\frac{1}{2}} \mathbf{L}_j \right)^{-1} \mathbf{L}_j^T D_{\mathbf{m}^*(\boldsymbol{\theta}_{b,0})}^{\frac{1}{2}} \mathbf{A}_{X_j}(\boldsymbol{\theta}_{b+1,0}), \end{aligned} \quad (29)$$

$$\mathbf{A}_{X_j}(\boldsymbol{\theta}_{b+1,0}) \equiv D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{\frac{1}{2}} \mathbf{X}_j \left(\mathbf{X}_j^T D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})} \mathbf{X}_j \right)^{-1} \mathbf{X}_j^T D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{\frac{1}{2}}, \quad (30)$$

for $j = b, b+1$. The asymptotic distribution of the test statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}|\hat{\boldsymbol{\theta}}_b^{\phi_2})$ will be a chi-square if the matrix \mathbf{T}_b^* is idempotent and symmetric. It is clear that \mathbf{T}_b^* is symmetric, we shall establish that it is also idempotent. Since $\mathcal{C}(D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{1/2} \mathbf{X}_{b+1}) \subset \mathcal{C}(D_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{1/2} \mathbf{X}_b)$ we have

$$\begin{aligned} \mathbf{A}_{X_{b+1}}(\boldsymbol{\theta}_{b+1,0}) &= \mathbf{A}_{X_{b+1}}(\boldsymbol{\theta}_{b+1,0}) \mathbf{A}_{X_{b+1}}(\boldsymbol{\theta}_{b+1,0}) = \mathbf{A}_{X_{b+1}}(\boldsymbol{\theta}_{b+1,0}) \mathbf{A}_{X_b}(\boldsymbol{\theta}_{b+1,0}), \\ \mathbf{A}_{X_b}(\boldsymbol{\theta}_{b+1,0}) &= \mathbf{A}_{X_b}(\boldsymbol{\theta}_{b+1,0}) \mathbf{A}_{X_b}(\boldsymbol{\theta}_{b+1,0}), \end{aligned}$$

and on the other hand since $\mathcal{C}(\mathbf{L}_b) \subset \mathcal{C}(\mathbf{L}_{b+1})$ there exists a matrix \mathbf{B} such that $\mathbf{L}_b = \mathbf{L}_{b+1} \mathbf{B}$. Thus it holds

- i) $\mathbf{K}_{b+1} = \mathbf{K}_{b+1} \mathbf{K}_{b+1} = \mathbf{K}_{b+1} \mathbf{K}_b = \mathbf{K}_b \mathbf{K}_{b+1}$,
- ii) $\mathbf{K}_b = \mathbf{K}_b \mathbf{K}_b$,

which implies $\mathbf{T}_b^* \mathbf{T}_b^* = \mathbf{T}_b^*$.

The degrees of freedom of the chi-squared distributed statistic $T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2}|\hat{\boldsymbol{\theta}}_b^{\phi_2})$ coincides with the trace of the matrix \mathbf{T}_b^* , i.e. $t_b - t_{b+1} - r_b + r_{b+1}$. \square

For the test statistic $S^{\phi}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi}|\hat{\boldsymbol{\theta}}_b^{\phi})$ the same result as Theorem 4.2 can be obtained by following a similar proof.

REMARK 4.2. *Consider the saturated LMLC, i.e. the design matrix of the loglinear model is given by a $k \times k$ matrix \mathbf{X}_1 ($t_1 = k$), and we may assume, without any loss*

of generality, $\mathbf{X}_1 = \mathbf{I}_k$. On the other hand, apart from the constraints associated with the sampling scheme ($c \geq 0$) there is no any additional linear constraint ($r_1 = 0$)

$$M_1 = \{\mathbf{m}(\boldsymbol{\theta}_1) \in \mathbb{R}^k : \log \mathbf{m}(\boldsymbol{\theta}_1) = \boldsymbol{\theta}_1, \boldsymbol{\theta}_1 \in \Theta_1\},$$

$$\Theta_1 = \{\boldsymbol{\theta}_1 \in \mathbb{R}^k : \mathbf{X}_0^T \mathbf{m}(\boldsymbol{\theta}_1) = \mathbf{X}_0^T \mathbf{n}\}.$$

Consider also a generic LMLC

$$M_2 = \{\mathbf{m}(\boldsymbol{\theta}_2) \in \mathbb{R}^k : \log \mathbf{m}(\boldsymbol{\theta}_2) = \mathbf{X}_2 \boldsymbol{\theta}_2, \boldsymbol{\theta}_2 \in \Theta_2\},$$

$$\Theta_2 = \{\boldsymbol{\theta}_2 \in \mathbb{R}^{t_2} : \mathbf{L}_2^T \mathbf{m}(\boldsymbol{\theta}_2) = \mathbf{d}_2\},$$

where $t_2 \leq k$, $r_2 \geq 0$, being strict at least one of the two inequalities. The hypothesis testing (24) for the two nested LMLC above ($B = 2$) is the same as the goodness-of-fit test (17) associated with the model M_2 . Therefore $T^{\phi_1}(\hat{\boldsymbol{\theta}}_2^{\phi_2} | \hat{\boldsymbol{\theta}}_1^{\phi_2}) = T^{\phi_1}(\hat{\boldsymbol{\theta}}_2^{\phi_2})$.

To test the sequence of LMLC (24) $b = 1, \dots, b^*$, we need an asymptotic independence result for the sequence of test statistics $\{T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \hat{\boldsymbol{\theta}}_b^{\phi_2})\}_{b=1}^{b^*}$ (or $\{S^{\phi}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi} | \hat{\boldsymbol{\theta}}_b^{\phi})\}_{b=1}^{b^*}$). This result is given in the theorem below.

THEOREM 4.3. *Suppose that data $\mathbf{n} = (n_1, \dots, n_k)^T$ are Poisson distributed. We first test, $H_{Null} : M_{b+1}$ against $H_{Alt} : M_b$, followed by $H_{Null} : M_b$ against $H_{Alt} : M_{b-1}$. Then, under the assumption that it holds M_{b+1} , the statistics $T^{\phi_1}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \hat{\boldsymbol{\theta}}_b^{\phi_2})$ and $T^{\phi_1}(\hat{\boldsymbol{\theta}}_b^{\phi_2} | \hat{\boldsymbol{\theta}}_{b-1}^{\phi_2})$ are asymptotically independent.*

Proof. A second order Taylor's expansion gives

$$T^{\phi_1}(\hat{\boldsymbol{\theta}}_j^{\phi_2} | \hat{\boldsymbol{\theta}}_{j-1}^{\phi_2}) = \tilde{\mathbf{Z}}_j^T \tilde{\mathbf{Z}}_j + o_P(1), \quad j \in \{b+1, b\},$$

where

$$\tilde{\mathbf{Z}}_j = \mathbf{T}_{j-1}^* \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \frac{1}{\sqrt{N}} (\mathbf{n} - \mathbf{m}(\boldsymbol{\theta}_{b+1,0})),$$

and $\mathbf{T}_j^* = \mathbf{K}_j - \mathbf{K}_{j+1}$ with \mathbf{K}_j , $j = b, b-1$ defined in (29). By Searle [30, Theorem 4 in page 59] the quadratic forms

$$\tilde{\mathbf{Z}}_j^T \tilde{\mathbf{Z}}_j = \frac{1}{\sqrt{N}} (\mathbf{n} - \mathbf{m}(\boldsymbol{\theta}_{b+1,0}))^T \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \mathbf{T}_{j-1}^* \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \frac{1}{\sqrt{N}} (\mathbf{n} - \mathbf{m}(\boldsymbol{\theta}_{b+1,0})),$$

for $j = b+1, b$, are asymptotically independent if

$$\mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \mathbf{T}_{b+1}^* \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \boldsymbol{\Sigma} \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} \mathbf{T}_b^* \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}^{-\frac{1}{2}} = \mathbf{0}_{k \times k},$$

where matrix $\boldsymbol{\Sigma} = \mathbf{D}_{\mathbf{m}^*(\boldsymbol{\theta}_{b+1,0})}$ is the asymptotic variance-covariance matrix of vector $(\mathbf{n} - \mathbf{m}(\boldsymbol{\theta}_{b+1,0}))/\sqrt{N}$. By following a similar argument given in the proof of Theorem 4.2 to see that \mathbf{T}_b^* is idempotent, it follows that $\mathbf{T}_{b+1}^* \mathbf{T}_b^* = \mathbf{0}_{k \times k}$. \square

For (27) the same result as Theorem 4.2 can be obtained by following a similar proof.

According to Agresti [1, page 215] once an asymptotic probability of error I equals $1 - (1 - \alpha)^{\frac{1}{B-1}}$ has been established for each test in a sequence of nested tests, the overall asymptotic probability of type I error is less or equal than α . In the next theorem a stronger result is given.

THEOREM 4.4. *For a sequence of $B-1$ tests (24) associated with a sequence of LMLC $\{M_b\}_{b=1}^B$, when each test has a size equals $1 - (1 - \alpha)^{\frac{1}{B-1}}$, the overall size of the tests is given by α .*

Proof. For the purpose of establishing a size equals $1 - (1 - \alpha)^{\frac{1}{B-1}}$ for each hypothesis testing in (21) we shall consider according to Theorem 4.2 a cutpoint for the rejection region equals $\chi_{df}^2((1 - \alpha)^{\frac{1}{B-1}})$, where $df = t_b - t_{b+1} - r_b + r_{b+1}$. Thus, the overall size for testing (24) in a sequence of nested LMLC $\{M_b\}_{b=1}^B$ is given by

$$\begin{aligned} & \Pr(\exists b \in \{1, \dots, B-1\} : T^{\phi_1}(\widehat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \widehat{\boldsymbol{\theta}}_b^{\phi_2}) > \chi_{df}^2((1 - \alpha)^{\frac{1}{B-1}}) | H_{Null}(b)) \\ &= 1 - \Pr(T^{\phi_1}(\widehat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \widehat{\boldsymbol{\theta}}_b^{\phi_2}) \leq \chi_{df}^2((1 - \alpha)^{\frac{1}{B-1}}) | H_{Null}(b), b = 1, \dots, B-1) \\ &= 1 - \prod_{b=1}^{B-1} \Pr(T^{\phi_1}(\widehat{\boldsymbol{\theta}}_{b+1}^{\phi_2} | \widehat{\boldsymbol{\theta}}_b^{\phi_2}) \leq \chi_{df}^2((1 - \alpha)^{\frac{1}{B-1}}) | H_{Null}(b)) \\ &= 1 - \prod_{b=1}^{B-1} \Pr(\chi_{df}^2 \leq \chi_{df}^2((1 - \alpha)^{\frac{1}{B-1}})) \\ &= 1 - \prod_{b=1}^{B-1} (1 - \alpha)^{\frac{1}{B-1}} = \alpha. \end{aligned}$$

The second equality comes from Theorem 4.3 and the third one from Theorem 4.2. \square

5 Simulation study: Marginal homogeneity

5.1 Description of conditional and unconditional tests

The traditionally so-called conditional test for marginal homogeneity (MH) was applied for the first time in Caussinus [5]. He noted that once it is known that the quasi-symmetry (QS) model holds, marginal homogeneity (MH) is equivalent to symmetry (S). In other words, because QS is a nested model within S ($M_{QS} \subset M_S$), first we could test whether it holds QS model against the alternative hypothesis of saturated model (SAT), defined in Remark 4.2 ($M_{QS} \subset M_{SAT}$),

$$H_{Null}(1) : M_{QS} \text{ against } H_{Alt}(1) : M_{SAT} - M_{QS}, \quad (31)$$

and after that

$$H_{Null}(2) : M_S \text{ against } H_{Alt}(2) : M_S - M_{QS}. \quad (32)$$

Focussed on a square $I \times I$ contingency table with multinomial sampling ($c = 1$), one could be interested in analyzing what the difference is between testing the conditional model above and the unconditional model of MH below

$$H_{Null} : M_{MH} \text{ against } H_{Alt} : M_{SAT} - M_{MH}. \quad (33)$$

The formulation of these models for two-way contingency tables is

$$\sum_{j=1}^I m_{ij}(\boldsymbol{\theta}_{MH}) = \sum_{i=1}^I m_{ij}(\boldsymbol{\theta}_{MH}) \text{ or } m_{i\bullet}(\boldsymbol{\theta}_{MH}) = m_{\bullet i}(\boldsymbol{\theta}_{MH}), \quad i, j = 1, \dots, I;$$

$$\log m_{ij}(\boldsymbol{\theta}_S) = u + \theta_i + \theta_j + \theta_{ij}, \quad i, j = 1, \dots, I,$$

with $\theta_{ij} = \theta_{ji}$, $\forall i \neq j$, $\sum_{i=1}^I \theta_i = 0$, $\sum_{i=1}^I \theta_{12(ij)} = 0$, $j = 1, \dots, I$;

$$\log m_{ij}(\boldsymbol{\theta}_{OQS}) = u + \theta_{1(i)} + \theta_{1(j)} + \beta w_j + \theta_{12(ij)}, \quad i, j = 1, \dots, I,$$

with $\theta_{12(ij)} = \theta_{12(ji)}$, $\forall i \neq j$, $\sum_{i=1}^I \theta_{1(i)} = 0$, $\sum_{i=1}^I \theta_{12(ij)} = 0$, $j = 1, \dots, I$, $\sum_{j=1}^I w_j = 0$, $\sum_{j=1}^I w_j^2 = 1$, where $\{w_j\}_{j=1}^I$ is a set of weights associated to each category $j \in \{1, \dots, I\}$

such that the distance between the contiguous ones is fixed, i.e.

$$w_j = \frac{j - \sum_{i=1}^I i / I}{\left(\sum_{i=1}^I i^2 - \frac{1}{I} \left(\sum_{i=1}^I i \right)^2 \right)^{1/2}} = \frac{2j - (I+1)}{\sqrt{\frac{1}{3}I(I-1)(I+1)}}.$$

(for more details about the interpretation of this model see Agresti and Kateri [17]);

$$\log m_{ij}(\boldsymbol{\theta}_{QS}) = u + \theta_{1(i)} + \theta_{2(j)} + \theta_{12(ij)}, \quad i, j = 1, \dots, I,$$

with $\theta_{12(ij)} = \theta_{12(ji)}$, $\forall i \neq j$, $\sum_{i=1}^I \theta_{1(i)} = \sum_{j=1}^I \theta_{2(j)} = 0$, $\sum_{i=1}^I \theta_{12(ij)} = 0$, $j = 1, \dots, I$.

While M_S and M_{QS} are loglinear models, M_{MH} is a marginal model (see Remark 2.1). By taking into account the meaning of both tests, the initial conditions are different, actually while a rejection of H_{Null} implies that MH is not accepted, a rejection of $H_{Null}(1)$ does not implies the same fact, i.e. even though QS is rejected a MH could hold. However, an acceptance of H_{Null} and $H_{Null}(2)$ implies the same hypothesis, i.e. MH is accepted. Because all models, M_{SAT} , M_{QS} , M_S , and M_{MH} , are LMLC, it is possible to establish the same true model for both tests (31)-(32) and (33), by choosing a suitable parametrization according to the design matrices. On the other hand, it will be possible to carry out such a test in order to compare the exact size and power, through a simulation experiment by using ϕ -divergence based test-statistics. For this purpose we shall focus on $M\phi_{(\lambda_2)}E$'s with $\lambda_2 \in \{-0.5, 0, 2/3, 1, 2\}$ (i.e., with $\lambda_2 = 0$ MLE's are included), as well as on the same family of $\phi_{(\lambda_1)}$ -divergence measures (7) for building test-statistics, with $\lambda_1 \in \{-0.5, 0, 2/3, 1, 2\}$ (i.e., once MLE's are included, with $\lambda_1 = 0$ and $\lambda_1 = 1$ the classical test-statistics are obtained, likelihood ratio $G^2(\hat{\boldsymbol{\theta}}_{b+1}|\hat{\boldsymbol{\theta}}_b)$ and chi-squared $X^2(\hat{\boldsymbol{\theta}}_{b+1}|\hat{\boldsymbol{\theta}}_b)$ test-statistics respectively)

$$T^{\phi_{(\lambda_1)}}(\hat{\boldsymbol{\theta}}_{b+1}^{\phi_{(\lambda_2)}}|\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}}) = \begin{cases} \frac{2}{\lambda_1(\lambda_1+1)} \left(\sum_{i=1}^I \sum_{j=1}^I \frac{m_{ij}^{\lambda_1+1}(\hat{\boldsymbol{\theta}}_{b-1}^{\phi_{(\lambda_2)}})}{m_{ij}^{\lambda_1}(\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}})} - n \right), & \lambda_1(\lambda_1+1) \neq 0 \\ 2 \sum_{i=1}^I \sum_{j=1}^I m_{ij}(\hat{\boldsymbol{\theta}}_{b-1}^{\phi_{(\lambda_2)}}) \log \frac{m_{ij}(\hat{\boldsymbol{\theta}}_{b-1}^{\phi_{(\lambda_2)}})}{m_{ij}(\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}})}, & \lambda_1 = 0 \\ 2 \sum_{i=1}^I \sum_{j=1}^I m_{ij}(\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}}) \log \frac{m_{ij}(\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}})}{m_{ij}(\hat{\boldsymbol{\theta}}_{b-1}^{\phi_{(\lambda_2)}})}, & \lambda_1 = -1 \end{cases}$$

where n and I are respectively the total table count and the table size (i.e., $k = I^2$ and $n = \sum_{i=1}^I \sum_{j=1}^I n_{ij}$). In particular, if the saturated LMLC is considered as M_b , then $m_{ij}(\hat{\boldsymbol{\theta}}_b^{\phi_{(\lambda_2)}}) = n_{ij}$ (see Remark 4.2).

We shall also consider another conditional test for MH, which is based on the ordinal quasi-symmetry model (OQS), instead of the previously considered QS,

$$H_{Null}(1') : M_{OQS} \text{ against } H_{Alt}(1') : M_{SAT} - M_{OQS}, \quad (34)$$

and

$$H_{Null}(2') : M_S \text{ against } H_{Alt}(2') : M_S - M_{OQS} \quad (35)$$

($M_S \subset M_{OQS} \subset M_{SAT}$). Although OQS is usually applied for ordinal categorical data, because its interpretation, it is possible to consider OQS as LMLC, in a generic way, by

defining its design matrix (for more information about this model see Agresti [2, Section 8.4]). In order to compute the powers for the two conditional tests, apart from considering a common true model, because $M_S \subset M_{\mathcal{OQS}} \subset M_{QS}$, we shall consider the same points of the alternative hypotheses.

TABLE 1
Theoretical probabilities for a $I \times I$ table ($I = 4$)

$m_{ij}^*(\theta_0)$	1	2	3	4	$m_{i\bullet}^*(\theta_0)$
1	0.08161	0.03156	0.01647	0.01050	0.14017
2	0.03156	0.21104	0.05204	0.01418	0.30883
3	0.01647	0.05204	0.22186	0.03156	0.32195
4	0.01050	0.01418	0.03156	0.17278	0.22905
$m_{\bullet j}^*(\theta_0)$	0.14017	0.30883	0.32195	0.22905	$m_{\bullet\bullet}^*(\theta_0) = 1$

In Table 1 the theoretical probability vector belonging to a multinomial sampling scheme with $n \in \{100, 250, 400, 550\}$ is shown. Its corresponding values for the parameters for each model (null hypotheses) are also given ($t_{\mathcal{MH}} = 16$, $t_S = 10$, $t_{\mathcal{OQS}} = 11$ and $t_{QS} = 13$):

$$\begin{aligned} \theta_{\mathcal{MH}} &= (u_{\mathcal{MH}}, \theta_{1(1)}, \theta_{1(2)}, \theta_{1(3)}, \theta_{2(1)}, \theta_{2(2)}, \theta_{2(3)}, \theta_{12(11)}, \theta_{12(12)}, \theta_{12(13)}, \\ &\quad \theta_{12(21)}, \theta_{12(22)}, \theta_{12(23)}, \theta_{12(31)}, \theta_{12(32)}, \theta_{12(33)})^T \\ &= (u_{\mathcal{MH}}, -0.95, -1.6, -2.05, -0.95, 0.95, -0.45, -1.75, -1.6, -0.45, \\ &\quad 1.0, -0.95, -2.05, -1.75, -0.95, 0.75)^T, \end{aligned}$$

$$\begin{aligned} \theta_S &= (u_S, \theta_1, \theta_2, \theta_3, \theta_{11}, \theta_{22}, \theta_{12}, \theta_{13}, \theta_{24}, \theta_{34})^T \\ &= (u_S, -0.35, 0.25, 0.3, 1.5, 1.25, -0.05, -0.75, -1, -0.25)^T, \end{aligned}$$

$$\theta_{\mathcal{OQS}} = (\theta_S, 0)^T, \theta_{QS} = (\theta_S, 0, 0, 0)^T$$

(the values of $u_{\mathcal{MH}}$ and u_S , are obtained through the different sampling sizes).

5.2 Simulated Exact sizes and powers

Focussing first on the tests (34)-(35), the simulation study is based on repeating the random experiments described above $R = 10,000$ times to compute on one hand the “exact sizes” by simulation

$$\begin{aligned} \alpha_n^{(\lambda_1, \lambda_2)}(\theta_{\mathcal{OQS}}) &= \frac{\#\{T^{\phi(\lambda_1)}(\hat{\theta}_{\mathcal{OQS}}^{\phi(\lambda_2)} | \hat{\theta}_{S\mathcal{AT}}^{\phi(\lambda_2)}) > \chi_{\frac{(I+1)(I-2)}{2}}^2((1-\alpha)^{\frac{1}{2}}) | \theta_{\mathcal{OQS}}\}}{R}, \\ \alpha_n^{(\lambda_1, \lambda_2)}(\theta_S) &= \frac{\#\{T^{\phi(\lambda_1)}(\hat{\theta}_S^{\phi(\lambda_2)} | \hat{\theta}_{\mathcal{OQS}}^{\phi(\lambda_2)}) > \chi_1^2((1-\alpha)^{\frac{1}{2}}) | \theta_S\}}{R}, \\ \alpha_n^{(\lambda_1, \lambda_2)} &= 1 - (1 - \alpha_n^{(\lambda_1, \lambda_2)}(\theta_{\mathcal{OQS}}))(1 - \alpha_n^{(\lambda_1, \lambda_2)}(\theta_S)), \end{aligned}$$

once a simulated “nominal” size of $1 - (1 - \alpha)^{\frac{1}{2}}$, with $\alpha = 0.05$, has been chosen for each test. In order to calculate simulated exact powers 12 points are chosen (6 for (34) and 6

for (35))

$$\begin{aligned}\boldsymbol{\theta}_{SAT}(i) &= (u_{SAT}, \theta_{1(1)}, \theta_{1(2)}, \theta_{1(3)}, \theta_{2(1)}, \theta_{2(2)}, \theta_{2(3)} + \delta_1(i), \theta_{12(11)}, \theta_{12(12)}, \\ &\quad \theta_{12(13)} + \delta_2(i), \theta_{12(21)}, \theta_{12(22)}, \theta_{12(23)}, \theta_{12(31)}, \theta_{12(32)}, \theta_{12(33)})^T \\ &= (u_{SAT}, -0.95, -1.6, -2.05, -0.95, 0.95, -0.45 + \delta_1(i), -1.75, -1.6, \\ &\quad -0.45 + \delta_2(i), 1.0, -0.95, -2.05, -1.75, -0.95, 0.75)^T,\end{aligned}\quad (36)$$

with $((\delta_1(1), \delta_2(1)), \dots, (\delta_1(6), \delta_2(6)))^T = ((0.45, 0), (0.7, 0), (0.9, 0), (0, 0.45), (0, 0.7), (0, 0.9))^T$.

$$\begin{aligned}\boldsymbol{\theta}_{OQS}(i) &= (u_{OQS}, \theta_1, \theta_2, \theta_3, \theta_{11}, \theta_{22}, \theta_{12}, \theta_{13}, \theta_{24}, \theta_{34}, \beta(i))^T \\ &= (u_{OQS}, -0.35, 0.25, 0.3, 1.5, 1.25, -0.05, -0.75, -1, -0.25, \beta(i))^T,\end{aligned}\quad (37)$$

with $(\beta(7), \dots, \beta(12))^T = (0.5, 0.7, 1.0, -0.5, -0.7, -1.0)^T$ (the values of u_{SAT} and u_{OQS} are obtained through the different sampling sizes). Thus the simulated exact powers are given by

$$\beta_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{SAT}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_{OQS}^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{SAT}^{\phi(\lambda_2)}) > \chi_{\frac{(I+1)(I-2)}{2}}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{SAT}(i)\}}{R},$$

$i = 1, \dots, 6$,

$$\beta_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{OQS}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_S^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{OQS}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{OQS}(i)\}}{R},$$

$i = 7, \dots, 12$. Focussing on the tests (31)-(32), simulated exact sizes are given by

$$\begin{aligned}\alpha_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{QS}) &= \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_{QS}^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{SAT}^{\phi(\lambda_2)}) > \chi_{\frac{(I-1)(I-2)}{2}}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{QS}\}}{R}, \\ \alpha_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_S) &= \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_S^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{QS}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_S\}}{R}, \\ \alpha_n^{(\lambda_1, \lambda_2)} &= 1 - (1 - \alpha_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{QS}))(1 - \alpha_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_S)),\end{aligned}$$

and to calculate simulated exact powers 12 points are chosen (6 for (31) and 6 for (32)), the same points (36) and (37) are valid taking into account that $\boldsymbol{\theta}_{QS}(i) = (\boldsymbol{\theta}_{OQS}(i), 0, 0)^T$,

$$\beta_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{SAT}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_{QS}^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{SAT}^{\phi(\lambda_2)}) > \chi_{\frac{(I-1)(I-2)}{2}}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{SAT}(i)\}}{R},$$

$i = 1, \dots, 6$,

$$\beta_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{QS}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_S^{\phi(\lambda_2)} | \hat{\boldsymbol{\theta}}_{QS}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{QS}(i)\}}{R},$$

$i = 7, \dots, 12$. For the goodness of fit test (31) we have, as usual,

$$\alpha_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{MH}) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_{MH}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{MH}\}}{R}$$

and to calculate simulated exact powers the same 12 points above are chosen

$$\beta_n^{(\lambda_1, \lambda_2)}(\boldsymbol{\theta}_{SAT}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\boldsymbol{\theta}}_{MH}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \boldsymbol{\theta}_{SAT}(i)\}}{R},$$

TABLE 2
 $\alpha_n^{(\lambda_1, \lambda_2)}$ with $n \in \{100, 250\}$

λ_2	λ_1	$n = 100$			$n = 250$		
		(33)	(31)-(32)	(34)-(35)	(33)	(31)-(32)	(34)-(35)
0	-0.5	0.0904	0.1973	0.2037	0.0541	0.1252	0.1086
	0	0.0694	0.0761	0.0686	0.0488	0.0646	0.0592
	2/3	0.0514	0.0455	0.0408	0.0443	0.0431	0.0450
	1	0.0479	0.0441	0.0401	0.0432	0.0391	0.0422
	2	0.0500	0.0707	0.0512	0.0441	0.0431	0.0434
2/3	-0.5	0.1407	0.2595	0.2600	0.0642	0.1374	0.1292
	0	0.0808	0.0807	0.0705	0.0523	0.0670	0.0603
	2/3	0.0376	0.0257	0.0244	0.0411	0.0372	0.0350
	1	0.0254	0.0188	0.0194	0.0369	0.0308	0.0295
	2	0.0118	0.0173	0.0159	0.0280	0.0214	0.0243
1	-0.5	0.1725	0.2872	0.2832	0.0743	0.1386	0.1500
	0	0.0923	0.0883	0.0780	0.0578	0.0658	0.0726
	2/3	0.0378	0.0257	0.0298	0.0422	0.0347	0.0371
	1	0.0237	0.0197	0.0258	0.0364	0.0288	0.0290
	2	0.0084	0.0181	0.0295	0.0239	0.0218	0.0179

TABLE 3
 $\alpha_n^{(\lambda_1, \lambda_2)}$ with $n \in \{400, 550\}$

λ_2	λ_1	$n = 400$			$n = 550$		
		(33)	(31)-(32)	(34)-(35)	(33)	(31)-(32)	(34)-(35)
0	-0.5	0.0587	0.0800	0.0704	0.0554	0.0664	0.0664
	0	0.0556	0.0589	0.0553	0.0530	0.0566	0.0566
	2/3	0.0526	0.0479	0.0482	0.0515	0.0495	0.0495
	1	0.0518	0.0454	0.0461	0.0510	0.0483	0.0483
	2	0.0520	0.0460	0.0464	0.0509	0.0483	0.0483
2/3	-0.5	0.0585	0.0877	0.0800	0.0578	0.0703	0.0703
	0	0.0503	0.0609	0.0555	0.0541	0.0585	0.0585
	2/3	0.0441	0.0440	0.0413	0.0494	0.0463	0.0463
	1	0.0415	0.0384	0.0372	0.0480	0.0418	0.0418
	2	0.0375	0.0299	0.0309	0.0433	0.0353	0.0353
1	-0.5	0.0628	0.0887	0.0960	0.0609	0.0738	0.0738
	0	0.0541	0.0581	0.0653	0.0559	0.0584	0.0584
	2/3	0.0444	0.0413	0.0439	0.0500	0.0457	0.0457
	1	0.0411	0.0367	0.0371	0.0479	0.0415	0.0415
	2	0.0344	0.0290	0.0256	0.0410	0.0335	0.0335

$i = 1, \dots, 6,$

$$\beta_n^{(\lambda_1, \lambda_2)}(\theta_{\mathcal{OQS}}(i)) = \frac{\#\{T^{\phi(\lambda_1)}(\hat{\theta}_{\mathcal{MH}}^{\phi(\lambda_2)}) > \chi_{I-1}^2((1-\alpha)^{\frac{1}{2}}) | \theta_{\mathcal{OQS}}(i)\}}{R},$$

$i = 7, \dots, 12.$ The results of simulated exact sizes are shown in Tables 2 and 3. To illustrate some representative values of the powers, in Table 4 the simulated exact powers focussed on the tests (34)-(35) are shown for the considered 12 points. The so-called *size corrected average gradient*, defined as

$$\gamma_n^{(\lambda_1, \lambda_2)} = \frac{\left(\frac{1}{12} \left(\sum_{i=1}^6 \left(\frac{\beta_n^{(\lambda_1, \lambda_2)}(i) - \alpha_n^{(\lambda_1, \lambda_2)}}{\delta_1(i) + \delta_2(i)} \right)^2 + \sum_{i=7}^{12} \left(\frac{\beta_n^{(\lambda_1, \lambda_2)}(i) - \alpha_n^{(\lambda_1, \lambda_2)}}{\beta(i)} \right)^2 \right) \right)^{\frac{1}{2}}}{\alpha_n^{(\lambda_1, \lambda_2)}},$$

is an overall measure of performance of the simulated exact size as well as the simulated exact powers (this measure was introduced for the first time in Rivas et al. [29]). Such a measure is interpreted a normalized mean rate of power gain with respect to the null hypothesis along the considered alternatives and it is therefore useful as criterion to select a test statistic (λ_1) as well as its estimator (λ_2) with the maximum value of $\gamma_n^{(\lambda_1, \lambda_2)}$. The values of $\gamma_n^{(\lambda_1, \lambda_2)}$ for the same kind of test-statistics considered in Tables 2 and 3 are shown in Tables 5 and 6.

5.3 Conclusions

A clear conclusion from the results of Tables 5 and 6 is the best performance of the sequence of tests (34)-(35) compared with (31)-(32), essentially because the tests (34)-(35) were much more powerful than (31)-(32). One possible reason, as it is explained in Agresti [1, page 373], could be that the power of a chi-square test tends to increase when degrees of freedom decrease. This explanation is valid for (35), however even a greater value of degrees of freedom of (34), the computed powers of the test (34) in this experiment were also greater than the powers of the test (31). In spite of that, we think that the last result could be affected by the common choice of (36), which means that the points within the region $M_{QS} - M_{\mathcal{OQS}}$ are excluded in the alternative hypothesis of (34) because these points fall within the points which are included in the null hypothesis of (31). On the other hand if one desires to compare the test-statistics associated with (34)-(35) with respect to ones of (33), even perhaps better performance of (34)-(35) there is no a big difference, and thus unless there is an evidence for thinking that there exists OQS before carrying out a MH test, it would be more convenient to use the unconditional HM test (33). Within the values of $\gamma_n^{(\lambda_1, \lambda_2)}$ for the same test, either (34)-(35) or (33), the variability is greater for (34)-(35) because the simulated exact sizes are also more variable. Apart from the criterion of the corrected average gradient, a criterion for excluding from the study the simulated exact sizes which are not close or fairly close to the nominal size ($\alpha = 0.05$) is necessary. Through the criterion given by Dale [11], the inequality

$$\left| \text{logit}(1 - \alpha_n^{(\lambda_1, \lambda_2)}) - \text{logit}(1 - \alpha) \right| \leq \epsilon,$$

with $\text{logit}(p) \equiv \log(p/(1-p))$, is considered, so that the two probabilities, $\alpha_n^{(\lambda_1, \lambda_2)}$ and α , are considered to be close if they satisfy such a inequality with $\epsilon = 0.35$ and fairly close if they satisfy with $\epsilon = 0.7$. Note that for $\alpha = 0.05$, $\epsilon = 0.35$ corresponds to $\alpha_n^{(\lambda_1, \lambda_2)} \in [0.0357, 0.0695]$ and $\epsilon = 0.7$ corresponds to $\alpha_n^{(\lambda_1, \lambda_2)} \in [0.0254, 0.0357] \cup (0.0695, 0.0954]$.

TABLE 4
 $\beta_n^{(\lambda_1, \lambda_2)}(i)$ with $\lambda_2 = \frac{2}{3}$ for tests (34)-(35)

n	λ_1	$i = 1$	$i = 4$	$i = 2$	$i = 5$	$i = 3$	$i = 6$
100	0	0.0720	0.0749	0.1139	0.1201	0.1749	0.1745
	2/3	0.0188	0.0163	0.0382	0.0389	0.0736	0.0745
	1	0.0110	0.0088	0.0251	0.0252	0.0522	0.0502
	2	0.0038	0.0035	0.0095	0.0111	0.0271	0.0257
250	0	0.0996	0.0989	0.2123	0.2091	0.3669	0.3637
	2/3	0.0582	0.0571	0.1522	0.1516	0.2956	0.2945
	1	0.0483	0.0468	0.1320	0.1330	0.2737	0.2716
	2	0.0337	0.0323	0.1063	0.1047	0.2303	0.2329
400	0	0.1251	0.1161	0.3162	0.3230	0.5575	0.5631
	2/3	0.0973	0.0882	0.2760	0.2837	0.5226	0.5249
	1	0.0859	0.0797	0.2616	0.2692	0.5072	0.5093
	2	0.0698	0.0639	0.2334	0.2427	0.4786	0.4840
550	0	0.1536	0.1453	0.4363	0.4410	0.7292	0.7347
	2/3	0.1337	0.1202	0.4072	0.4102	0.7072	0.7178
	1	0.1258	0.1096	0.3942	0.3984	0.6995	0.7074
	2	0.1095	0.0941	0.3734	0.3757	0.6832	0.6911
n	λ_1	$i = 7$	$i = 10$	$i = 8$	$i = 11$	$i = 9$	$i = 12$
100	0	0.0678	0.0724	0.1197	0.1370	0.2419	0.2880
	2/3	0.0572	0.0634	0.1050	0.1206	0.2183	0.2646
	1	0.0527	0.0597	0.0993	0.1148	0.2088	0.2562
	2	0.0511	0.0550	0.0931	0.1061	0.1949	0.2452
250	0	0.1798	0.1980	0.3597	0.3901	0.6588	0.7164
	2/3	0.1704	0.1861	0.3477	0.3739	0.6432	0.7010
	1	0.1661	0.1812	0.3418	0.3677	0.6363	0.6955
	2	0.1592	0.1750	0.3311	0.3557	0.6252	0.6859
400	0	0.3112	0.3369	0.5785	0.6124	0.8739	0.9190
	2/3	0.3037	0.3258	0.5683	0.6041	0.8685	0.9153
	1	0.3000	0.3229	0.5636	0.6009	0.8664	0.9137
	2	0.2917	0.3156	0.5544	0.5935	0.8610	0.9098
550	0	0.4249	0.4578	0.7272	0.7734	0.9630	0.9806
	2/3	0.4177	0.4510	0.7215	0.7671	0.9607	0.9798
	1	0.4147	0.4480	0.7191	0.7651	0.9598	0.9797
	2	0.4077	0.4431	0.7117	0.7593	0.9581	0.9789

TABLE 5
 $\gamma_n^{(\lambda_1, \lambda_2)}$ with $n \in \{100, 250\}$

λ_2	λ_1	$n = 100$			$n = 250$		
		(33)	(31)-(32)	(34)-(35)	(33)	(31)-(32)	(34)-(35)
0	0	1.9498	0.8109	1.8522	8.8777	2.2680	6.5096
	2/3	2.3191	0.8173	3.3489	9.5346	3.4695	8.5937
	1	2.3989	0.8030	3.3222	9.7044	3.8527	9.2039
	2	2.3323	0.7436	2.5181	9.4949	3.5113	9.0488
2/3	0	2.2460	0.2623	1.4553	8.4591	2.0100	6.5121
	2/3	3.3642	1.2907	4.5181	10.0525	3.4108	11.1638
	1	4.1761	1.8014	5.5197	10.8193	3.9618	13.1137
	2	6.1807	1.6047	6.4614	13.1578	5.3177	15.5109
1	0	2.0916	0.8894	1.2676	7.8486	1.7541	5.5833
	2/3	3.3562	0.8595	3.3604	9.8390	3.3627	10.5545
	1	4.2558	0.9214	3.7848	10.8838	4.1772	12.5275
	2	7.1583	1.0689	3.0949	14.5192	6.2040	15.9077

TABLE 6
 $\gamma_n^{(\lambda_1, \lambda_2)}$ with $n \in \{400, 550\}$

λ_2	λ_1	$n = 400$			$n = 550$		
		(33)	(31)-(32)	(34)-(35)	(33)	(31)-(32)	(34)-(35)
0	0	11.5475	5.1013	10.5516	15.1129	7.5141	13.4170
	2/3	12.0922	6.3045	12.1017	15.4706	8.6282	14.9108
	1	12.2447	6.6720	12.6871	15.5996	8.8334	15.0571
	2	12.1903	6.6276	12.6977	15.6361	8.8885	14.9610
2/3	0	13.0615	4.8961	10.9942	14.8684	7.3124	13.7563
	2/3	14.4991	6.3327	14.7299	16.0838	8.8237	17.1009
	1	15.2220	7.1010	16.2884	16.4405	9.5901	18.3816
	2	16.3217	8.7403	19.3697	18.0044	10.9238	21.2854
1	0	12.2295	4.3335	10.1186	14.4705	6.8205	12.7432
	2/3	14.4409	6.3769	14.1149	15.8989	8.8589	16.2596
	1	15.3240	7.4787	15.7674	16.4489	10.0339	17.8500
	2	17.4356	10.5383	19.5188	18.8438	12.4853	21.8208

Those simulated exact sizes which are taken as close according to the Dale's criterion have been marked in blue color in Tables 1 and 2, and in red color fairly close simulated exact sizes. Finally, it is concluded that the best overall choice for the test statistics $T^{\phi(\lambda_1)}(\widehat{\boldsymbol{\theta}}^{\phi(\lambda_2)})$ is $(\lambda_1, \lambda_2) \in \{(1, 1), (1, \frac{2}{3})\}$, however for the smallest sample size ($n = 100$) the test statistic associated with $\lambda_1 = 1$ is not a good choice and it is better $(\lambda_1, \lambda_2) \in \{(\frac{2}{3}, 1), (\frac{2}{3}, \frac{2}{3})\}$.

To finalize, we would like to comment that LMLC can have been dealt in this paper in a more general setting by following generalized log-linear models, $C \log(A\mathbf{m}(\boldsymbol{\theta})) = \mathbf{X}\boldsymbol{\theta}$ (see Lang [18] and references therein). With these models it would be possible to consider loglinear constraints for the marginal distributions by considering $C = \mathbf{I}_k$ and $A \neq \mathbf{I}_k$. Furthermore, using minimum power divergence estimators a different type of application for $\log(A\mathbf{m}(\boldsymbol{\theta})) = \mathbf{X}\boldsymbol{\theta}$, with $A \neq \mathbf{I}_k$ and Poisson sampling, can be found in Martín and Li [23].

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